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Short-Time Gibbsianness for Infinite-Dimensional Diffusions with Space-Time Interaction

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Abstract We consider a class of infinite-dimensional diffusions where the interaction between the components has a finite extent both in space and time. We start the system from a Gibbs measure with a finite-range uniformly bounded interaction. Under suitable conditions on the drift, we prove that there exists $t_0 > 0$ such that the distribution at time $t \leq t_0$ is a Gibbs measure with absolutely summable interaction. The main tool is a cluster expansion of both the initial interaction and certain time-reversed Girsanov factors coming from the dynamics.

Keywords Infinite-dimensional diffusion · Cluster expansion · Time-reversal · Non-Markovian drift · Girsanov formula · Delay equations

1 Introduction

In this paper we study short-time Gibbsianness for a class of infinite-dimensional diffusions with general space-time interaction. The diffusion $X = (X_i(t))_{t\geq 0, i\in\mathbb{Z}^d}$ is the solution of the

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following stochastic differential equations (SDE)

$$dX_i(t) = b_i(t, X)\,dt + dB_i(t), \quad t > 0,\ i \in \mathbb{Z}^d \tag{1}$$

with values in $\mathbb{R}^{\mathbb{Z}^d}$ and starting at time 0 in a Gibbs measure called ν. The drift term b which characterizes the type of interaction between the coordinates is adapted but can be non-Markovian, i.e. the interaction at time t may depend on the values of X on the full time interval $[0, t]$, so-called delay term. Stochastic delay equations are relevant due to their various applications for example in biomathematics and finance (see e.g. [15] and [9]).

Earlier short-time Gibbsianness results could be obtained for b defined as a gradient of a Hamiltonian, see [1]. In that case one even has a Gibbs structure on the path space $C(\mathbb{R}_+, \mathbb{R})^{\mathbb{Z}^d}$ and the existence of a reversible stationary measure for the dynamics, properties which are heavily used by the authors in their proof. In contrast, in our model, as soon as the drift is not of gradient form, we do not know if a reversible measure exists (in fact, even the existence of a stationary measure is not guaranteed). Another important difference with previous works consists in the fact that, since the form of the interaction between the spins (the drift) is quite general, one cannot make use any more of a decoupling method: this tool was used in [1] to compare the infinite-dimensional dynamics with another simpler dynamics, where a spin is forced to be independent of the others.

For discrete Ising spins there exist also short-time results; for example, in [3, 13], Gibbsianness is proved for general local dynamics, making use of a reversible measure. See also [4, 10, 11] for results in this direction for non-discrete bounded spins. The idea behind short-time conservation of Gibbsianness is that the time-evolved measure is in a certain sense close to the initial one, which was assumed to be Gibbs. In the case of discrete spins following a dynamics of Glauber type (see [13]), this means that there is a "sea" (in the sense of percolation) of spins that did not change at all, and isolated islands (for which there is a Peierls estimate) of spins where at least one flip occurred. This picture is implemented in a cluster expansion of the Radon-Nikodym derivative of the finite-volume distribution at time t w.r.t. the finite-volume distribution at time 0. In order to obtain Gibbsianness, one has to show that the sum of the cluster weights containing a fixed site (the origin) is finite, uniformly in Λ, for t small enough. The cluster weights have contributions from the interaction in the initial measure and from Girsanov factors coming from the dynamics. The Girsanov factors are multiplicative functionals which are close to 1 for small t, and therefore in good shape for a cluster expansion.

We prove here that for general drifts satisfying assumptions (A1)–(A3) the law of the infinite-dimensional diffusion (1) stays Gibbsian for a (short) time. In the case of continuous unbounded spins, the picture is similar as for discrete spins but technically much more involved. One has to set up a cluster expansion of the Radon-Nikodym derivative of the finite-volume measure at time $t > 0$ too. The factor coming from the Girsanov formula now contains stochastic integrals, which cannot be turned into ordinary integrals as is done in the reversible case, using Itô's formula. The control of the Girsanov factors reduces to a control of exponential moments of time-reversal of these stochastic integrals, which can be done under some regularity conditions. Furthermore, our results lead, as a corollary, to a constructive local existence for a class of infinite-dimensional SDE with non-Markovian drift (see [2] also).

The rest of our paper is organized as follows. In Sect. 2 we give definitions of Gibbs measures, assumptions on the dynamics, as well as some non-trivial examples satisfying the assumptions (A1)–(A3). In Sect. 3 we state and prove our main theorem. In Appendix we come back to the examples and verify the assumptions for them.

2 Notations and Definitions

In this section we want to define the necessary framework for our study. In particular we present a class of examples to which our results apply.

2.1 Interactions and Gibbs Measures

We will work with measures on the configuration space $\mathbb{R}^{\mathbb{Z}^d}$. Elements of $\mathbb{R}^{\mathbb{Z}^d}$ are denoted x, y, z. For $\Lambda \subset \mathbb{Z}^d$, $x, y \in \mathbb{R}^{\mathbb{Z}^d}$, we denote $x_\Lambda z_{\Lambda^c}$ the configuration obtained by concatenating the restriction of x to Λ with the restriction of z to Λ^c, i.e.,

$$(x_\Lambda z_{\Lambda^c})_i = \begin{cases} x_i & \text{if } i \in \Lambda, \\ z_i & \text{if } i \in \Lambda^c. \end{cases} \tag{2}$$

We choose the initial distribution ν to be a *Gibbs measure* associated to an interaction φ and an a priori measure m. Let us recall some definitions.

Definition 2.1 An interaction φ on $\mathbb{R}^{\mathbb{Z}^d}$ is a collection of functions φ_Λ from $\mathbb{R}^{\mathbb{Z}^d}$ to $\mathbb{R}$, where Λ is any finite subset of $\mathbb{Z}^d$, which satisfy following properties

1. φ_Λ is $\mathcal{F}_\Lambda$-measurable, where $\mathcal{F}_\Lambda$ is the sigma-field generated by the canonical projection on $\mathbb{R}^\Lambda$.
2. φ is absolutely summable, i.e., for all $i \in \mathbb{Z}^d$,
$$\sum_{\Lambda \ni i} \|\varphi_\Lambda\|_\infty < \infty.$$
3. Translation invariance:
$$\varphi_{\Lambda+i}(\tau_i x) = \varphi_\Lambda(x)$$
where τ_i denotes the shift over i: $(\tau_i x)_j := x_{j-i}$.

Furthermore, we assume in this paper just as in [1] that the initial interaction φ is

(a) of finite range, i.e., there exists a $r > 0$ such that if $diam(\Lambda) > r \Rightarrow \varphi_\Lambda \equiv 0$
(b) $\forall \Lambda$, φ_Λ is Lipschitz continuous.

Given the interaction φ we define the associated *Hamiltonian* $h = (h_\Lambda)_{\Lambda \subset \mathbb{Z}^d}$ with respect to the boundary condition $z \in \mathbb{R}^{\mathbb{Z}^d}$ by

$$h_\Lambda(x_\Lambda, z_{\Lambda^c}) = \sum_{\Lambda' : \Lambda' \cap \Lambda \neq \emptyset} \varphi_{\Lambda'}(x_\Lambda z_{\Lambda^c}). \tag{3}$$

(The above sum is finite since φ is of finite range.)

The *finite-volume Gibbs measure* with boundary condition z w.r.t. an a priori measure m is then given by

$$\nu_{\Lambda,z}(dx_\Lambda) = \frac{1}{Z_\Lambda^z} \exp(-h_\Lambda(x_\Lambda, z_{\Lambda^c}))\, m(dx_\Lambda) \tag{4}$$

where $Z_\Lambda^z = \int_{\mathbb{R}^\Lambda} \exp(-h_\Lambda(y_\Lambda, z_{\Lambda^c})) m(dy_\Lambda)$ is the finite-volume partition function. We consider as a priori measure m a finite product measure, absolutely continuous w.r.t. Lebesgue measure. By absolute summability of the interactions, we then have that the partition functions Z_Λ^z are finite.

As usual, the *finite-volume Gibbs measure with free boundary conditions* is defined by

$$\nu_\Lambda(dx_\Lambda) := \frac{1}{Z_\Lambda} \exp\Big(-\sum_{A\subset\Lambda} \varphi_A(x_\Lambda)\Big) m(dx_\Lambda). \tag{5}$$

Definition 2.2 The measure μ is a Gibbs measure with interaction φ and a priori measure m, if for all finite $\Lambda \subset \mathbb{Z}^d$ and smooth test functions f $\mathcal{F}_\Lambda$-measurable, the so-called DLR conditions are satisfied:

$$\int f(x_\Lambda)\mu(dx) = \iint f(x_\Lambda)\nu_{\Lambda,z}(dx_\Lambda)\mu(dz). \tag{6}$$

This means that $\nu_{\Lambda,z}$ is a version of the conditional probability $\mu(dx_\Lambda | x_{\Lambda^c} = z_{\Lambda^c})$.

Let us recall that (6) is satisfied for all Λ as soon as it is satisfied for singletons (see for example [5]), i.e., as soon as for each $i \in \mathbb{Z}^d$

$$\mu(dx_i | x_{\mathbb{Z}^d\setminus i}) = \frac{\exp(-h_i(x_i, x_{\mathbb{Z}^d\setminus i}))}{\int \exp(-h_i(y_i, x_{\mathbb{Z}^d\setminus i}))m(dy_i)} m(dx_i) \tag{7}$$

where h_i is given by (3) for $\Lambda = \{i\}$.

Remark 2.1 Since we restrict ourselves in this paper to interactions which are uniformly bounded, certain natural interactions such as quadratic ones are not included here. In fact, all "unboundedness" is hidden in the a priori measure (the log of the density of m has to be unbounded for the partition functions to be finite). In [12] unbounded interactions of a specific type are considered for a dynamics of independent Ornstein-Uhlenbeck processes.

2.2 Dynamics

Let $\Omega = C(\mathbb{R}_+, \mathbb{R})^{\mathbb{Z}^d}$ be the path space for continuous trajectories of the time evolution of the continuous spin system endowed with the canonical sigma-field $\mathcal{F}$.

We denote by $P = \otimes_{i\in\mathbb{Z}^d} P_i$ the Wiener measure on Ω, resp. by $P^x = \otimes_{i\in\mathbb{Z}^d} P_i^{x_i}$ the Wiener measure with deterministic initial condition $x = (x_i)_i \in \mathbb{R}^{\mathbb{Z}^d}$, which will be denoted in the finite-volume case just by $P_\Lambda^x = \otimes_{i\in\Lambda} P_i^{x_i}$.

Moreover $P_{[0,t]}^{x,y}$ is the law of a Brownian bridge on $[0, t]$ obtained by conditioning P to be at time 0 in x and at time t in y.

The time-reversal $\theta = (\theta_t)_{t>0}$ is a family of functionals on the path space Ω. It is defined as follows for $t > 0$ and $\omega(\cdot) \in C([0, t], \mathbb{R})^{\mathbb{Z}^d}$:

$$\theta_t\omega(\cdot) := \omega(t - \cdot). \tag{8}$$

We consider the following infinite-dimensional system of stochastic differential equations:

$$\begin{cases} dX_i(t) = b_i(t, X)\, dt + dB_i(t), & t > 0,\ i \in \mathbb{Z}^d, \\ X(0) \sim \nu \end{cases} \tag{9}$$

where $(B_i)_{i\in\mathbb{Z}^d}$ is a sequence of real-valued independent Brownian motions. The drift term $b_i(t, \omega)$ at time t *may possibly* depend on the values of ω on the whole time interval $[0, t]$, thus in particular X could be non-Markovian. We suppose the existence of a solution of the system (9) and denote it by Q^ν, resp. Q^x if the initial condition is deterministic ($\nu = \delta_x$).

The drift term $b = (b_i)_i$ satisfies the following assumptions (A1)–(A3).

(A1) *Translation invariant, finite range and adapted*, i.e.,

$$\forall i \in \mathbb{Z}^d, \quad b_i(t,\omega) = b_0(t, \tau_i \omega) \tag{10}$$

$$\text{and} \quad b_0(t,\omega) = b_0(t, \omega_{\mathcal{N}}) = b_0(t, (\omega_{\mathcal{N}}(s) : 0 \le s \le t)) \tag{11}$$

where $\mathcal{N} \subset \mathbb{Z}^d$ is a fixed finite connected set containing the origin.

(A2) *Lipschitz continuous* uniformly on each compact time interval: $\forall T > 0\ \exists K(T) > 0$ such that for all $0 \le t \le T$,

$$|b_0(t,\omega) - b_0(t,\omega')| \le K(T) \sup_{0 \le s \le t,\, j \in \mathcal{N}} |\omega_j(s) - \omega'_j(s)| \tag{12}$$

$$\text{and} \quad |b_0(t,0)| \le K(T). \tag{13}$$

(A3) *Exponential moment of some time-reversal functional*: b is such that, if F_0^t denotes the functional

$$F_0^t(X) := \int_0^t b_0(s,X)dX_0(s) - \frac{1}{2}\int_0^t b_0^2(s,X)ds \tag{14}$$

its time-reversal $F_0^t \circ \theta_t$ is well defined and satisfies

$$\lim_{t \to 0} \mathbb{E}_{P^x}\left(\left[\exp\left(|F_0^t \circ \theta_t|\right) - 1\right]^{2p}\right) = 0 \tag{15}$$

for every initial condition $x \in \mathbb{R}^{\mathbb{Z}^d}$. Here, the integer p is the next odd number larger than $\max(\mathit{range}(b), \mathit{range}(\varphi)) + 1$ (see the proof of Lemma 3.5, (61)).

We want to present some classes of drifts b which satisfy the above requirements (A1)–(A3).

Example 2.1 (Markovian Drift) Let $b_0(t,\omega) = b_0(t, \omega_{\mathcal{N}}(t))$ be a Lipschitz continuous Markovian drift with finite range $\mathcal{N}$. Moreover we assume for $j \in \mathcal{N}$ the existence of the first derivative $b'_0 := \frac{\partial}{\partial x_0} b_0$ with, for all $T > 0$

$$\|b\|_{T,\infty} + \|b'\|_{T,\infty} := \sup_{t \le T} \sup_{x \in \mathbb{R}^{\mathbb{Z}^d}} \left(|b_0(t,x)| + |b'_0(t,x)|\right) < +\infty. \tag{16}$$

This class encloses in particular the Hamiltonian drift from Theorem 1 in [1] as a special case.

The second example describes an interaction between the coordinates which is spatially degenerate (self-interaction) but has long temporal memory and is thus non-Markovian.

Example 2.2 (Long Memory Case) Let the drift b_i be defined as

$$b_i(t,\omega) = \int_0^t \epsilon(s)(\omega_i(s) - \omega_i(0))ds \tag{17}$$

where the locally integrable *memory function* $\epsilon : [0, \infty] \to \mathbb{R}$ has the weak continuity property

$$\int_0^t \epsilon(s)ds \underset{t\to 0}{\longrightarrow} 0. \tag{18}$$

The third example is a generalisation of the first ones.

Example 2.3 (Interaction with Finite Extent in Space and Time) Let b be given by

$$b_i(t,\omega) = \int_0^t \alpha_i(t-s, \omega(s)-\omega(0))dV_s \tag{19}$$

where the integrator V_s can be deterministic or stochastic (adapted) and of bounded variation. The functions α_i are Lipschitz continuous and spatially local, i.e.,

$$\alpha_i(t-s,x) = \alpha_0(t-s,(\tau_i x)_{\mathcal{N}}). \tag{20}$$

The proof that the drifts described in the three examples above satisfy assumptions (A1)–(A3) is postponed to the Appendix. In particular we will there explicitly compute the time-reversal functional $F_0^t \circ \theta_t$ and provide a proof of the existence of its exponential moments.

3 Main Result and Its Proof

The following theorem is the main result of our paper.

Theorem 3.1 *Let us consider an infinite-dimensional Brownian diffusion solution of the system of SDE* (9) *where the drift term b satisfies the properties* (A1)–(A3) *given in the previous section. Suppose the initial distribution* ν *to be a Gibbs measure associated to an a priori measure m and to a finite-range Lipschitz continuous interaction* φ.

Then there is a time $t_0 := t_0(\varphi, b) > 0$ *such that for any time* $t \le t_0$ *the law of the diffusion at time t is a Gibbs measure associated to the a priori measure m and to an absolutely summable interaction* φ^t.

A main step in the proof of Theorem 3.1 is a representation lemma, presented in the next subsection.

3.1 The Finite-Dimensional Density at Time t

Let us first introduce a finite-volume dynamics in Λ in the following way

$$\begin{cases} dX_i^\Lambda(t) = b_i(t,X)dt + dB_i(t), & i\in\Lambda \text{ such that } \mathcal{N}+i\subseteq\Lambda, \\ dX_i^\Lambda(t) = dB_i(t), & i\in\Lambda \text{ such that } \mathcal{N}+i\nsubseteq\Lambda. \end{cases} \tag{21}$$

Its existence (and uniqueness) is ensured by the assumption (A2) (see Theorem 11.2 in [16]). For the finite-volume Gibbs measure with free boundary conditions ν_Λ we denote by ν_Λ^t the distribution of $(X_i^\Lambda(t))_{i\in\Lambda}$ starting from ν_Λ at time 0. Note that the law of the path $(X_i^\Lambda(.))_{i\in\Lambda}$ is absolutely continuous with respect to the Wiener measure on any finite time interval (see e.g. [14], Theorem 7.2). Therefore, at any fixed time t, ν_Λ^t—as ν_Λ—has a density w.r.t. $m(dx_\Lambda)$, which we denote by $f_\Lambda^t(x_\Lambda)$.

Lemma 3.1 *Let $f^t_\Lambda(x_\Lambda)$ be the density of the finite-volume probability measure ν^t_Λ w.r.t. $m(dx_\Lambda)$. Let therefore f^0_Λ denote the (Gibbsian) density of ν_Λ w.r.t. $m(dx_\Lambda)$. Then there exists a time $t_0 > 0$, such that for any $t \le t_0$ the ratio f^t_Λ/f^0_Λ admits a cluster representation*:

$$\frac{d\nu^t_\Lambda}{d\nu_\Lambda}(x_\Lambda) = \frac{f^t_\Lambda(x_\Lambda)}{f^0_\Lambda(x_\Lambda)} = \exp\left(\sum_{\Gamma\subset\Lambda} a(\Gamma) w^t(\Gamma, x_\Gamma)\right) \tag{22}$$

where the "cluster weights" $w^t(\Gamma, x_\Gamma)$ satisfy

$$\forall i \in \mathbb{Z}^d, \sum_{\Gamma\ni i} \sup_{x\in\mathbb{R}^{\mathbb{Z}^d}} |w^t(\Gamma, x_\Gamma)| < \infty \tag{23}$$

and $a(\Gamma)$ are combinatorial factors. The sum runs over clusters Γ which will be described in (74).

Suppose Lemma 3.1 holds true. We now show why it implies the claim of the main theorem when $\nu = \nu^{free}$, a so-called Gibbs measure with free boundary conditions, that is a limit of finite-volume Gibbs measures with free boundary conditions.

Let us denote by $\Upsilon^{t,i}_\Lambda(x)$ the conditional density

$$\Upsilon^{t,i}_\Lambda(x) := \frac{f^t_\Lambda(x_\Lambda)}{\int f^t_\Lambda(y_i x_{\Lambda\setminus i}) m(dy_i)} \tag{24}$$

and rewrite it as

$$\Upsilon^{t,i}_\Lambda(x) = \frac{f^t_\Lambda(x_\Lambda)}{f^0_\Lambda(x_\Lambda)} \times \left[\int \frac{f^t_\Lambda(y_i x_{\Lambda\setminus i})}{f^0_\Lambda(y_i x_{\Lambda\setminus i})} \frac{f^0_\Lambda(y_i x_{\Lambda\setminus i})}{f^0_\Lambda(x_i x_{\Lambda\setminus i})} m(dy_i)\right]^{-1}. \tag{25}$$

Then using the claim of Lemma 3.1 we have a cluster representation of the ratio f^t_Λ/f^0_Λ. So (25) becomes

$$\begin{aligned}
&\exp\left(\sum_{\Gamma\subset\Lambda} a(\Gamma) w^t(\Gamma, x_\Gamma)\right) \\
&\quad\times\left[\int \exp\left(\sum_{\Gamma\subset\Lambda} a(\Gamma) w^t(\Gamma, (y_i x_{\Lambda\setminus i})_\Gamma)\right)\right. \\
&\quad\left.\times \exp\left(-\sum_{\substack{\Lambda'\subset\Lambda\\ \Lambda'\ni i}} \varphi_{\Lambda'}(y_i x_{\Lambda\setminus i}) - \varphi_{\Lambda'}(x_i x_{\Lambda\setminus i})\right) m(dy_i)\right]^{-1}
\end{aligned} \tag{26}$$

where we used (5) to express the integral in the r.h.s. of (25) in terms of the interaction φ. The sum runs over all clusters Γ whose support is contained in the subset Λ. Since the sum over all clusters whose support does not contain i cancels out in the ratio, (26) becomes

$$\Upsilon^{t,i}_\Lambda(x) = \frac{\exp(\sum_{\substack{\Gamma\subset\Lambda\\ \Gamma\ni i}} a(\Gamma) w^t(\Gamma, x_\Gamma) - \sum_{\substack{\Lambda'\subset\Lambda\\ \Lambda'\ni i}} \varphi_{\Lambda'}(x_\Lambda))}{\int \exp(\sum_{\substack{\Gamma\subset\Lambda\\ \Gamma\ni i}} a(\Gamma) w^t(\Gamma, (y_i x_{\Lambda\setminus i})_\Gamma) - \sum_{\substack{\Lambda'\subset\Lambda\\ \Lambda'\ni i}} \varphi_{\Lambda'}(y_i x_{\Lambda\setminus i})) m(dy_i)}. \tag{27}$$

Due to the claim of Lemma 3.1, $\Upsilon^{t,i}_\Lambda(x)$ converges, as Λ goes to $\mathbb{Z}^d$, uniformly in x, towards

$$\frac{\exp(-h^t_i(x_i, x_{\mathbb{Z}^d\setminus i}))}{\int \exp(-h^t_i(y_i, x_{\mathbb{Z}^d\setminus i})) dm(y_i)} \tag{28}$$

where h_i^t is given by

$$h_i^t(x_i, x_{\mathbb{Z}^d\setminus i}) = -\sum_{\Gamma\ni i} a(\Gamma) w^t(\Gamma, x_\Gamma) + \sum_{\Lambda'\ni i} \varphi_{\Lambda'}(x) \tag{29}$$

which is built from an absolutely summable interaction φ^t. In particular we have proven that uniformly in x, for $t \le t_0$

$$d\nu_\Lambda^t(x_i|x_{\Lambda\setminus i}) \underset{\Lambda\to\mathbb{Z}^d}{\longrightarrow} d\nu^t(x_i|x_{\mathbb{Z}^d\setminus i}) \tag{30}$$

which implies that ν^t is Gibbs, since ν_Λ^t converges weakly to ν_t, and hence the uniform limit of $\nu_\Lambda^t(dx_i|x_{\Lambda\setminus i})$ is a continuous version of the conditional distribution $\nu^t(dx_i|x_{\mathbb{Z}^d\setminus i})$, see Lemma 3.2 below.

Lemma 3.2 *Let $(\mu_\Lambda)_\Lambda$ be a collection of probability measures on $\mathbb{R}^{\mathbb{Z}^d}$ indexed by finite subsets $\Lambda \subset \mathbb{Z}^d$. Suppose that $\mu_\Lambda \to \mu$ weakly as $\Lambda \uparrow \mathbb{Z}^d$. Suppose that the conditional distributions satisfy*

$$\lim_{\Lambda\uparrow\mathbb{Z}^d} \mu_\Lambda(dx_i|x_{\Lambda\setminus\{i\}}) = \overline{\mu}(dx_i, x_{\mathbb{Z}^d\setminus\{i\}}) \tag{31}$$

where the limit is in the weak sense as a measure on dx_i, and is uniform in the variable $x_{\mathbb{Z}^d\setminus\{i\}}$. Then $\overline{\mu}(dx_i, x_{\mathbb{Z}^d\setminus\{i\}})$ is a continuous version of the family of conditional probabilities $\mu(dx_i|x_{\mathbb{Z}^d\setminus\{i\}})$.

Proof By uniform convergence, $x \mapsto \overline{\mu}(dx_i, x_{\mathbb{Z}^d\setminus\{i\}})$ is weakly continuous (as a map from $\mathbb{R}^{\mathbb{Z}^d}$ into the set of probability measures on $\mathbb{R}$), so it suffices to show that it is a version of the conditional probability $\mu(dx_i|x_{\mathbb{Z}^d\setminus\{i\}})$, or equivalently that for every local and bounded continuous function $f : \mathbb{R}^{\mathbb{Z}^d} \to \mathbb{R}$

$$\int f(x_i x_{\mathbb{Z}^d\setminus\{i\}})\mu(dx) = \iint f(x_i x_{\mathbb{Z}^d\setminus\{i\}})\overline{\mu}(dx_i, x_{\mathbb{Z}^d\setminus\{i\}})\mu(dx_{\mathbb{Z}^d\setminus\{i\}}). \tag{32}$$

By locality of f, for $\Lambda \subset \mathbb{Z}^d$ large enough,

$$\begin{aligned}\int f d\mu_\Lambda &= \iint f(x_i x_{\Lambda\setminus\{i\}})\mu_\Lambda(dx_i|x_{\Lambda\setminus\{i\}})\mu_\Lambda(dx_{\Lambda\setminus\{i\}}) \\ &= \iint f(x_i x_{\mathbb{Z}^d\setminus\{i\}})\mu_\Lambda(dx_i|x_{\Lambda\setminus\{i\}})\mu_\Lambda(dx_{\mathbb{Z}^d\setminus\{i\}}).\end{aligned} \tag{33}$$

On one hand, by the weak convergence,

$$\lim_{\Lambda\uparrow\mathbb{Z}^d} \int f d\mu_\Lambda = \int f d\mu.$$

On the other hand, by the uniform convergence (31), the right-hand side of (33) equals

$$\iint f(x_i x_{\mathbb{Z}^d\setminus\{i\}})\overline{\mu}(dx_i, x_{\mathbb{Z}^d\setminus\{i\}})\mu_\Lambda(dx_{\mathbb{Z}^d\setminus\{i\}}) + \epsilon_\Lambda$$

where ϵ_Λ goes to zero as $\Lambda \uparrow \mathbb{Z}^d$. Since the map

$$x \mapsto \int f(x_i x_{\mathbb{Z}^d\setminus\{i\}})\overline{\mu}(dx_i, x_{\mathbb{Z}^d\setminus\{i\}})$$

is bounded and continuous, by weak convergence of μ_Λ to μ, we have

$$\lim_{\Lambda\uparrow\mathbb{Z}^d}\iint f(x_i x_{\mathbb{Z}^d\setminus\{i\}})\overline{\mu}(dx_i, x_{\mathbb{Z}^d\setminus\{i\}})\mu_\Lambda(dx_{\mathbb{Z}^d\setminus\{i\}})$$
$$=\iint f(x_i x_{\mathbb{Z}^d\setminus\{i\}})\overline{\mu}(dx_i, x_{\mathbb{Z}^d\setminus\{i\}})\mu(dx_{\mathbb{Z}^d\setminus\{i\}}). \tag{34}$$

Combining both convergences, it implies (32). □

Let us remark that in order to prove (22) in the previous lemma we can replace the reference measure m by any other one, for example by the Lebesgue measure since

$$\frac{d\nu_\Lambda^t}{dm}\Big/\frac{d\nu_\Lambda}{dm}=\frac{d\nu_\Lambda^t}{dx}\Big/\frac{d\nu_\Lambda}{dx}. \tag{35}$$

3.2 Cluster Expansion of the Finite-Dimensional Density

Let ν_Λ^t be the finite-volume time-evolved measure with initial free boundary conditions defined above. To prove Lemma 3.1 we perform a cluster expansion of $\frac{d\nu_\Lambda^t}{dx}/\frac{d\nu_\Lambda}{dx}$.

To do this, we first provide a representation of f_Λ^t , the density of ν_Λ^t w.r.t. Lebesgue measure. By the Lebesgue density theorem, the density f of an absolutely continuous measure μ w.r.t. the Lebesgue measure can be computed via

$$f(x)=\lim_{\varepsilon\to 0}\int\frac{h_\varepsilon(y)}{2\varepsilon}\mu(dy) \tag{36}$$

where $h_\varepsilon(y)=\mathbb{1}_{[x-\varepsilon,x+\varepsilon]}(y)$.

In a $|\Lambda|$-dimensional situation, one takes $h_\varepsilon(x_\Lambda)=\prod_{i\in\Lambda}h_\varepsilon^i(x_i)$;

Thus, for $\mu=\nu_\Lambda^t$

$$\int\frac{1}{2\varepsilon}h_\varepsilon(x_\Lambda)d\nu_\Lambda^t(x_\Lambda)=\mathbb{E}_{Q_\Lambda^{\nu_\Lambda}}\left(\frac{1}{2\varepsilon}h_\varepsilon(X(t))\right)=\int\frac{1}{2\varepsilon}\mathbb{E}_{Q_\Lambda^{y_\Lambda}}(h_\varepsilon(X(t)))\nu_\Lambda(dy_\Lambda). \tag{37}$$

This leads to

Lemma 3.3 *The density f_Λ^t of ν_Λ^t w.r.t. the Lebesgue measure is given by*

$$f_\Lambda^t(x_\Lambda)=\int_{\mathbb{R}^\Lambda}\mathbb{E}_{P^{y,x}_{[0,t],\Lambda}}\left(\prod_{i\in\Lambda}\exp\left(\int_0^t b_i(s,X)dX_i(s)-\frac{1}{2}\int_0^t b_i^2(s,X)ds\right)\right)$$
$$\times\prod_{i\in\Lambda}p_t(y_i,x_i)\nu_\Lambda(dy_\Lambda) \tag{38}$$

where p_t is the transition kernel of a standard Brownian motion and $P^{y,x}_{[0,t],\Lambda}$ is the law of the $|\Lambda|$-dimensional Brownian bridge being at time 0 in y_Λ and in x_Λ at time t.

Proof Using Girsanov's formula

$$dQ_\Lambda^y(X)=\exp\left(\sum_{i\in\Lambda}\int_0^t b_i(s,X)dX_i(s)-\frac{1}{2}\int_0^t b_i^2(s,X)ds\right)dP_\Lambda^y(X) \tag{39}$$

where P^y_Λ is the product of independent Wiener measures. Then

$$\mathbb{E}_{Q^y_\Lambda}(h_\varepsilon(X(t))) = \mathbb{E}_{P^y_\Lambda}\left(\prod_{i\in\Lambda}\exp\left(\int_0^t b_i(s,X)dX_i(s) - \frac{1}{2}\int_0^t b_i^2(s,X)ds\right)h^i_\varepsilon(X_i(t))\right) \tag{40}$$

and taking the limit $\varepsilon \to 0$ gives

$$\lim_{\varepsilon\to 0}\frac{1}{(2\varepsilon)^{|\Lambda|}}\mathbb{E}_{Q^y_\Lambda}(h_\varepsilon(X_\Lambda(t))) = \mathbb{E}_{P^y_\Lambda}\left(\prod_{i\in\Lambda}\exp\left(\int_0^t b_i(s,X)dX_i(s) - \frac{1}{2}\int_0^t b_i^2(s,X)ds\right)\Big| X_\Lambda(t) = x_\Lambda\right) \times \prod_{i\in\Lambda} p_t(y_i,x_i) \tag{41}$$

which we can rewrite as

$$\mathbb{E}_{P^{y,x}_{[0,t],\Lambda}}\left(\prod_{i\in\Lambda}\exp\left(\int_0^t b_i(s,X)dX_i(s) - \frac{1}{2}\int_0^t b_i^2(s,X)ds\right)\right)\prod_{i\in\Lambda} p_t(y_i,x_i). \tag{42}$$

So finally we obtain, plugging in (42) into (37):

$$f^t_\Lambda(x_\Lambda) = \int_{\mathbb{R}^\Lambda}\mathbb{E}_{P^{y,x}_{[0,t],\Lambda}}\left(\prod_{i\in\Lambda}\exp\left(\int_0^t b_i(s,X)dX_i(s) - \frac{1}{2}\int_0^t b_i^2(s,X)ds\right)\right) \times \prod_{i\in\Lambda} p_t(y_i,x_i)\nu_\Lambda(dy_\Lambda). \tag{43}$$

□

Using the previous Lemma 3.3 we write the ratio f^t_Λ/f^0_Λ as

$$\frac{f^t_\Lambda(x_\Lambda)}{f^0_\Lambda(x_\Lambda)} = \int_{\mathbb{R}^\Lambda}\mathbb{E}_{P^{y,x}_{[0,t],\Lambda}}\left(\prod_{i\in\Lambda}\exp\left(\int_0^t b_i(s,X)dX_i(s) - \frac{1}{2}\int_0^t b_i^2(s,X)ds\right)\right) \times \prod_{i\in\Lambda} p_t(y_i,x_i)\exp\left(-\sum_{A\subset\Lambda}\varphi_A(y)\right)\exp\left(+\sum_{A\subset\Lambda}\varphi_A(x)\right)dy_\Lambda. \tag{44}$$

We will now prove

Lemma 3.4

$$\frac{f^t_\Lambda(x_\Lambda)}{f^0_\Lambda(x_\Lambda)} = \mathbb{E}_{P^x_\Lambda}(R^t_\Lambda\circ\theta_t), \tag{45}$$

where the functional R^t_Λ *is defined as*

$$R^t_\Lambda(X) := \prod_{i\in\Lambda}\exp\left(\int_0^t b_i(s,X)dX_i(s) - \frac{1}{2}\int_0^t b_i^2(s,X)ds\right) \times \exp\left(-\sum_{A\subset\Lambda}\varphi_A(X(0)) - \varphi_A(X(t))\right). \tag{46}$$

Proof Due to (46) and (44)

$$\frac{f^t_\Lambda(x_\Lambda)}{f^0_\Lambda(x_\Lambda)} = \int_{\mathbb{R}^\Lambda} \mathbb{E}_{P^{y,x}_{[0,t],\Lambda}}\big(R^t_\Lambda\big)\prod_{i\in\Lambda} p_t(y_i, x_i)dy_\Lambda. \tag{47}$$

Since $\theta_t \circ \theta_t = Id$ we can also write the ratio as

$$\int_{\mathbb{R}^\Lambda} \mathbb{E}_{P^{y,x}_{[0,t],\Lambda}}\big(R^t_\Lambda \circ \theta_t \circ \theta_t\big)\prod_{i\in\Lambda} p_t(y_i, x_i)dy_\Lambda \tag{48}$$

or

$$\int_{\mathbb{R}^\Lambda} \mathbb{E}_{P^{y,x}_{[0,t],\Lambda}\circ\theta_t^{-1}}\big(R^t_\Lambda \circ \theta_t\big)\prod_{i\in\Lambda} p_t(y_i, x_i)dy_\Lambda. \tag{49}$$

The image of the time-reversal of the Brownian bridge is again a Brownian bridge now with reversed starting and final points, $P^{y,x}_{[0,t],\Lambda} \circ \theta_t^{-1} = P^{x,y}_{[0,t],\Lambda}$. Furthermore the kernel p_t is symmetric, i.e., $p_t(x, y) = p_t(y, x)$. Thus the expectation is now taken w.r.t. a Brownian bridge starting in x and being in y at time t. Integrating out all possible final points y, the above integral (49) reduces simply to

$$\mathbb{E}_{P^x_\Lambda}(R^t_\Lambda \circ \theta_t) \tag{50}$$

which leaves us with an expectation w.r.t. independent Brownian motion starting in x_Λ of some time-reversed functional $R^t_\Lambda \circ \theta_t$. □

3.3 Cluster Estimates of $f^t_\Lambda / f^0_\Lambda$

To decompose the expectation (50) in terms of clusters we write $R^t_\Lambda(X)\circ\theta_t$ under the form $\exp -\sum_{A\subset\Lambda}\Psi_A(t,X)$ on the path space, where Ψ_A is $\mathcal{F}_A$-measurable and apply a standard Mayer expansion for t small. Ψ_A includes a contribution from Girsanov terms and the interaction at time 0.

Indeed one can write

$$R^t_\Lambda \circ \theta_t(X) = \prod_{A\subset\Lambda} \exp(-\Psi_A(t, X)) \tag{51}$$

with Ψ is defined as

$$\Psi_A(t, X) = \Phi^t_A(X) + \varphi_A(X(t)) - \varphi_A(X(0)) \tag{52}$$

and

$$\begin{cases} \Phi^t_{\mathcal{N}+i}(X) = -F^t_i \circ \theta_t(X) \quad (F^t_i(X) := F^t_0(X_{\cdot+i})), \\ \Phi^t_A(X) \equiv 0 \quad \text{if there does not exist } i \text{ such that } A = \mathcal{N} + i. \end{cases} \tag{53}$$

Next we give the usual definitions for performing a cluster expansion. Remember that the drift b and the initial interaction φ are of finite range. So we can fix a natural number $N = N(b, \varphi)$ which depends only on the range of b and φ such that for $|A| > N$, $\Psi_A \equiv 0$.

We call a cluster $\gamma = \{A_1, \dots, A_n\}$ a collection of such elements A_i such that any two $A_i, A_j \in \gamma$ are connected, i.e., there exists a sequence $i = i_1, \dots, i_m = j$ such that $A_{i_1} \cap A_{i_2} \neq \emptyset, \dots, A_{i_{m-1}} \cap A_{i_m} \neq \emptyset$. The support of the cluster γ is the finite subset $\bigcup_{i=1,\dots,n} A_i$ and is denoted by supp(γ). $|\gamma|$ denotes the cardinality of the support of γ. Clusters γ_i, γ_j are said compatible if their supports are disjoint. Let $\mathcal{C}_\Lambda$ be the set of all collections of compatible clusters in Λ. We expand

$$\prod_{A\subset\Lambda}\big(e^{-\Psi_A(t,X)} - 1 + 1\big) = 1 + \sum_{n=1}^{\infty} \sum_{\{\gamma_1,\dots,\gamma_n\}\in\mathcal{C}_\Lambda} \frac{1}{n!}\mathcal{K}^t(\gamma_1)(X)\cdots\mathcal{K}^t(\gamma_n)(X) \tag{54}$$

where

$$\mathcal{K}^t(\gamma)(X) = \prod_{A\in\gamma}\big(e^{-\Psi_A(t,X)} - 1\big). \tag{55}$$

Hence, following Lemma 3.4, we obtain

$$\begin{aligned}
f^t_\Lambda/f^0_\Lambda(x_\Lambda) &= \mathbb{E}_{P^x_\Lambda}\bigg(\prod_{A\subset\Lambda}\big(e^{-\Psi_A(t,X)} - 1 + 1\big)\bigg)\\
&= 1 + \mathbb{E}_{P^x_\Lambda}\bigg(\sum_{n=1}^{\infty} \sum_{\{\gamma_1,\dots,\gamma_n\}\in\mathcal{C}_\Lambda} \frac{1}{n!}\mathcal{K}^t(\gamma_1)(X)\cdots\mathcal{K}^t(\gamma_n)(X)\bigg)\\
&= 1 + \sum_{n=1}^{\infty} \sum_{\{\gamma_1,\dots,\gamma_n\}\in\mathcal{C}_\Lambda} \frac{1}{n!}\mathbb{E}_{P^x}\big(\mathcal{K}^t(\gamma_1)(X)\big)\cdots\mathbb{E}_{P^x}\big(\mathcal{K}^t(\gamma_n)(X)\big)\\
&=: 1 + \sum_{n=1}^{\infty} \sum_{\{\gamma_1,\dots,\gamma_n\}\in\mathcal{C}_\Lambda} \frac{1}{n!} w^t(\gamma_1, x)\cdots w^t(\gamma_n, x)
\end{aligned} \tag{56}$$

where the cluster weights are given by

$$w^t(\gamma, x) := \mathbb{E}_{P^x}\big(\mathcal{K}^t(\gamma)(X)\big). \tag{57}$$

We bound the weights w^t as follows.

Lemma 3.5 *There exists a strictly positive function $\lambda(t)$ which tends to* 0 *for $t \to 0$, such that for all clusters γ in Λ,*

$$\sup_{\Lambda,x} |w^t(\gamma, x)| \le e^{-c(t)|\gamma|} \tag{58}$$

where $c(t) := -\log(\lambda(t))$.

Proof The next technical problem is to interchange several times integration and products. We thus use the following generalised Hölder inequalities proved in Lemma 5.2 of [6].

Lemma 3.6 *Let $(\mu_k)_{k\in\chi}$ be a family of probability measures, each one defined on a space E_k where the indices k belong to a finite set χ. Let us also define a finite family $(g_i)_i$ of* *functions on* $E_\chi = \times_{k\in\chi} E_k$ *such that each* g_i *is* χ_i*-local for a certain* $\chi_i \subset \chi$ *in the sense that*

$$g_i(e) = g_i(e_{\chi_i}), \quad \text{for } e = (e_k)_{k\in\chi} \in E_\chi. \tag{59}$$

Let $p_i > 1$ *be numbers such that*

$$\forall k \in \chi, \quad \sum_{\{i:\chi_i \ni k\}} 1/p_i \le 1.$$

Then

$$\left|\int_{E_\chi} \prod_i g_i \otimes_{k\in\chi} d\mu_k\right| \le \prod_i \left(\int_{E_{\chi_i}} |g_i|^{p_i} \otimes_{k\in\chi_i} d\mu_k\right)^{1/p_i}. \tag{60}$$

We apply this lemma with $\chi = \mathrm{supp}(\gamma)$ ($\gamma =: \{A_1,\dots,A_n\}$), $\chi_i = A_i$, $g_i = e^{-\Psi_{A_i}} - 1$ and $\mu_k = P_k^x$. Let $p > N$ be the next odd number larger than N and let $p_i = p$ for all i. Then $\sum_{A_i\ni k} 1/p_i \le N/p \le 1$. Lemma 3.6 provides

$$|w^t(\gamma,x)| = \left|\mathbb{E}_{P^x}\left(\mathcal{K}^t(\gamma)(X)\right)\right| \le \prod_{i=1}^n \mathbb{E}_{P^x_{A_i}}\left(\left|e^{-\Psi_{A_i}(t,X)} - 1\right|^p\right)^{1/p}. \tag{61}$$

Recall that the functional Ψ_A was defined in (52). Since φ_A is Lipschitz continuous with a constant $C > 0$ independent of A, the cardinality of A_i is uniformly bounded by N and $\Phi_A^t \neq 0$ only if there exists a k such that $A = \mathcal{N} + k$, we obtain

$$|\Psi_A(t,X)| \le \mathbb{1}_{A=\mathcal{N}+k}|\Phi^t_{\mathcal{N}+k}(X)| + C\sup_{j\in A}|X_j(t) - X_j(0)|. \tag{62}$$

Using the simple fact that, for $a, b \ge 0$,

$$(e^b\cdot e^a - 1)^p \le 2^p\left(e^{p\cdot a}(e^b-1)^p + (e^a-1)^p\right) \tag{63}$$

and the estimate (62), we get

$$\begin{aligned}
&\mathbb{E}_{P^x}\left(\left|e^{-\Psi_A(t,X)} - 1\right|^p\right)\\
&\quad\le \mathbb{E}_{P^x}\left(\left(\exp\left(|\Phi^t_{\mathcal{N}+k}(X)|\mathbb{1}_{A=\mathcal{N}+k} + C\sup_{j\in A}|X_j(t)-X_j(0)|\right) - 1\right)^p\right)\\
&\quad\le 2^p\mathbb{E}_{P^x}\left(\exp\left(pC\sup_{j\in A}|X_j(t)-X_j(0)|\right)\left(\exp(|\Phi^t_{\mathcal{N}+k}(X)|\mathbb{1}_{A=\mathcal{N}+k}) - 1\right)^p\right)\\
&\qquad + 2^p\mathbb{E}_{P^x}\left(\left(\exp\left(C\sup_{j\in A}|X_j(t)-X_j(0)|\right) - 1\right)^p\right)\\
&\quad =: 2^p\mathbb{E}_{P^x}(G_{1,A}(t,X)) + 2^p\mathbb{E}_{P^x}(G_{2,A}(t,X)).
\end{aligned} \tag{64}$$

By the Cauchy-Schwarz inequality

$$\begin{aligned}
&\mathbb{E}_{P^x}(G_{1,A}(t,X))\\
&\quad\le \mathbb{E}_{P^x}\left(\left[\exp\left(|\Phi^t_{\mathcal{N}+k}(X)|\mathbb{1}_{A=\mathcal{N}+k}\right) - 1\right]^{2p}\right)^{1/2}
\end{aligned}$$

$$\times\, \mathbb{E}_{P^x}\big(\exp(2pC\sup_{j\in A}|X_j(t)-X_j(0)|)\big)^{1/2}$$

$$\le \mathbb{E}_{P^x}\big(\big(\exp|F_k^t\circ\theta_t(X)|-1\big)^{2p}\big)^{1/2}$$

$$\times\, \mathbb{E}_{P^x}\big(\exp\big(2pC\sup_{j\in A}|X_j(t)-X_j(0)|\big)\big)^{1/2}. \tag{65}$$

The exponential moment condition (A3) assures that $\exp|F_k^t(X)\circ\theta_t|$ converges in $L^{2p}(P^x)$ towards 1 for t going to 0 uniformly in x. So there exists a positive function $c_1(t)$ only depending on t vanishing when t is going to 0, such that

$$\mathbb{E}_{P^x}\big(\big(\exp|F_k^t\circ\theta_t(X)|-1\big)^{2p}\big)^{1/2} =: c_1(t). \tag{66}$$

The second term in (65) will be controlled as follows. We recall that X is a family of independent Brownian motions under P^x, thus

$$\mathbb{E}_{P^x}\big(\exp\big(2pC\sup_{j\in A}|X_j(t)-X_j(0)|\big)\big) \le \mathbb{E}(\exp(2pCN\sqrt{t}|Z|)) =: \overline{c}_1(t) \tag{67}$$

were Z is a standard Gaussian variable. Clearly, the function $\overline{c}_1(t)$ tends to 1 as t goes to 0. We now obtain,

$$\mathbb{E}_{P^x}(G_{1,A}(t,X)) \le c_1(t)\overline{c}_1(t) := C_1(t) \quad \text{with } \lim_{t\to 0} C_1(t)=0. \tag{68}$$

In a similar way we obtain

$$\begin{aligned}
\mathbb{E}_{P^x}(G_2(t,p,X)) &= \mathbb{E}_{P^x}\big(\big(\exp\big(C\sup_{j\in A}|X_j(t)-X_j(0)|\big)-1\big)^p\big) \\
&\le \mathbb{E}\big(\big(\exp\big(CN(b,\varphi)\sqrt{t}|Z|\big)-1\big)^p\big) \\
&\le \mathbb{E}\left(\left(\int_0^{CN\sqrt{t}|Z|}\exp(u)du\right)^p\right) \\
&\le (CN)^p\sqrt{t}^p\,\mathbb{E}\big(|Z|^p\exp(pCN\sqrt{t}|Z|)\big) =: c_2(t),
\end{aligned} \tag{69}$$

where $c_2(t)$ vanishes for t small. So finally

$$\mathbb{E}_{P^x}(G_2(t,p,X)) \le c_2(t) \quad \text{with } \lim_{t\to 0} c_2(t)=0. \tag{70}$$

Thus, calling

$$\lambda(t) := 2\big(C_1(t)+c_2(t)\big)^{1/Np} \quad \text{and} \quad c(t) = -\log\lambda(t) \tag{71}$$

we obtain the desired cluster weight bound

$$|w^t(\gamma,x)| = \big|\mathbb{E}_{P^x}\big(\mathcal{K}^t(\gamma)(X)\big)\big| \le \exp(-c(t)|\gamma|). \tag{72}$$

Note that this bound is uniform in the initial condition x.

To complete the proof of Lemma 3.1, we need a cluster expansion of $\log(\frac{f_\Lambda^t(x_\Lambda)}{f_\Lambda^0(x_\Lambda)})$. This will be done using the Kotecký-Preiss criterion (see [8], p. 492). The bound (72) provides that, for $t \le t_0$ small enough and any $\gamma \subset \Lambda$,

$$\sup_{x\in\mathbb{R}^{\mathbb{Z}^d}} \sup_{\Lambda\subset\mathbb{Z}^d} \sum_{\gamma':\mathrm{supp}(\gamma)\cap\mathrm{supp}(\gamma')\neq\emptyset} |w^t(\gamma',x)|e^{|\gamma'|} \le |\gamma|. \tag{73}$$

Indeed, by the finite-range assumption, the number of clusters γ of size n, containing a fixed point is bounded by e^{cn} where $c > 0$ does not depend on t. So an absolutely convergent expansion of the logarithm of the series (56) exists for t small enough:

$$\begin{aligned}\log\left(\frac{f^t_\Lambda(x_\Lambda)}{f^0_\Lambda(x_\Lambda)}\right) &= \sum_{n=1}^{\infty} \sum_{\Gamma:=\{\gamma_1,\dots,\gamma_n\}\in U_\Lambda} a(\gamma_1,\dots,\gamma_n)w^t(\gamma_1,x)\cdots w^t(\gamma_n,x)\\ &=: \sum_{\Gamma\subset\Lambda} a(\Gamma)w^t(\Gamma,x)\end{aligned} \tag{74}$$

with $a(\Gamma)$ and $a(\gamma_1,\dots,\gamma_n)$ purely combinatorial terms coming from the Taylor expansion, and $w^t(\Gamma,x)$ depends only on x_Γ. The set U_Λ is the set of all compatible clusters whose union is connected too, the latter sum runs over all clusters Γ which consist of compatible γ_i. The proof of Lemma 3.1 is now completed. □

Next we want to show that if the diffusion starts with any Gibbs measure ν, i.e., not necessarily with a measure with free boundary conditions ν^{free}, the probability measure ν^t is Gibbs associated to the same interaction.

To do so, use the well-known fact (see e.g. [5]) that every Gibbs measure associated to a given interaction φ is a mixture of extremal Gibbs measures, which are themselves limits of finite-volume Gibbs measures with fixed boundary conditions. Now, fix a boundary condition z and look at the finite-volume dynamics (21) where the initial distribution is given by $\nu_{\Lambda,z}$ instead of ν_Λ. We call $\nu^t_{\Lambda,z}$ the distribution of $(X_i^\Lambda(t))_{i\in\Lambda}$ starting from $\nu_{\Lambda,z}$. One can without difficulty adapt the result of Lemma 3.1 to the case with this boundary condition. There exists a similar cluster expansion, with weights (depending on z) which can be controlled too. Now the main argument is the following: The upper bounds in (62) are uniform in z since the Lipschitz constant C of the interaction is independent of the boundary condition. Therefore the bounds on the cluster weights—similar to (72)—are uniform in z, and the cluster expansion-generalizing (74)—converges when the volume Λ goes to $\mathbb{Z}^d$.

Moreover, if we start from a Gibbs measure ν which is translation-invariant, we can show that ν_t remains Gibbs using another way, i.e. the variational principle characterizing Gibbs measures (see [5], Sect. 15.4). This applies in our context, even if spins are unbounded, since the a priori measure is finite and the interactions are absolutely summable. This argument of Gibbsianness via the variational principle has the advantage that the implication "ν_t Gibbs for some translation-invariant initial Gibbs measure ν implies ν_t Gibbs for all translation-invariant initial Gibbs measures" is true even for t large and is itself not related to the method of cluster expansions. The drawback is however that one has to restrict to translation-invariant Gibbs measures.

First notice that, if initially the relative entropy density $i(\nu|\nu^{free})$ vanishes, then the relative entropy density of the time-evolved measure satisfies

$$i(\nu^t|\nu^{t,free}) \le i(\nu|\nu^{free}) = 0$$

for all $t \ge 0$. Hence if $\nu^{t,free}$ is Gibbs with a absolutely summable interaction, then ν^t is Gibbs with the same interaction. Notice that this fact does not depend on t being small.

In the lemma below we show that $i(\nu|\nu^{free})=0$ is zero for every extremal Gibbs measure ν with interaction φ. By convexity of the relative entropy density, this then extends to all Gibbs measures ν with interaction φ. The proof follows the standard argument of the variational principle (boundary condition independence of the pressure), see [5]. We prefer to spell it out however, for the sake of completeness, as we are in a context of unbounded spins.

Lemma 3.7 *Let ν be a translation-invariant Gibbs measure and ν^{free} a Gibbs measure with free boundary conditions. Then the relative entropy density $i(\nu|\nu^{free})$ vanishes.*

Proof Let $\Lambda\subset\mathbb{Z}^d$, ν and ν^{free} be defined as in the assumption of the lemma. The relative entropy in volume Λ of ν w.r.t. ν_Λ^{free} is defined by

$$I_\Lambda(\nu_\Lambda|\nu_\Lambda^{free})=\int\log\left(\frac{d\nu_\Lambda}{d\nu_\Lambda^{free}}(x_\Lambda)\right)\nu_\Lambda(dx_\Lambda). \tag{75}$$

By the DLR conditions,

$$d\nu_\Lambda(x_\Lambda)=\int\frac{\exp(-h_\Lambda(x_\Lambda,z_{\Lambda^c}))}{Z_\Lambda^z}\nu_{\Lambda^c}(dz_{\Lambda^c}). \tag{76}$$

As usual, we show that

$$\frac{d\nu_\Lambda}{d\nu_\Lambda^{free}}(x_\Lambda)\le\exp(o(|\Lambda|)) \tag{77}$$

uniformly in x, where

$$\frac{d\nu_\Lambda}{d\nu_\Lambda^{free}}(x_\Lambda)=\int\exp\left(-\sum_{\substack{A\cap\Lambda^c\neq\emptyset\\ A\cap\Lambda\neq\emptyset}}\varphi_A(x_\Lambda z_{\Lambda^c})\right)\frac{Z_\Lambda^{free}}{Z_\Lambda^z}\nu_\Lambda(dz_{\Lambda^c}). \tag{78}$$

The ratio of the partition functions is equal to

$$\begin{aligned}
\frac{Z_\Lambda^{free}}{Z_\Lambda^z}&=\frac{1}{Z_\Lambda^z}\int\exp(-h_\Lambda(x_\Lambda))m(dx_\Lambda) && (79)\\
&=\int\exp(-h_\Lambda(x_\Lambda)+h_\Lambda(x_\Lambda,z_\Lambda))\frac{\exp(-h_\Lambda(x_\Lambda,z_\Lambda))}{Z_\Lambda^z}m(dx_\Lambda) && (80)\\
&=\int\exp\left(\sum_{\substack{A\cap\Lambda\neq\emptyset\\ A\cap\Lambda^c\neq\emptyset}}\varphi_A(x_\Lambda z_{\Lambda^c})\right)\frac{\exp(-h_\Lambda(x_\Lambda,z_{\Lambda^c}))}{Z_\Lambda^z}m(dx_\Lambda) && (81)\\
&=\mathbb{E}_{\nu_\Lambda}\left(\exp\left(\sum_{\substack{A\cap\Lambda\neq\emptyset\\ A\cap\Lambda^c\neq\emptyset}}\varphi_A(X_\Lambda z_{\Lambda^c})\right)\right). && (82)
\end{aligned}$$

We bound the interaction by its supnorm and use that φ is absolute summable to deduce that

$$\sum_{\substack{A\cap\Lambda\neq\emptyset\\ A\cap\Lambda^c\neq\emptyset}}\varphi_A(x_\Lambda z_{\Lambda^c})\le o(|\Lambda|) \tag{83}$$

which means for the ratio of the partition functions (79) that

$$\frac{Z_{\Lambda}^{free}}{Z_{\Lambda}^{z}} \leq \exp(o(|\Lambda|)) \tag{84}$$

and a fortiori we conclude that

$$\frac{d\nu_{\Lambda}}{d\nu_{\Lambda}^{free}}(x_{\Lambda}) \leq \int \exp(o(|\Lambda|))\nu_{\Lambda}(dz_{\Lambda^c}) \tag{85}$$

$$= \exp(o(|\Lambda|)). \tag{86}$$

The relative entropy becomes

$$I_{\Lambda}(\nu_{\Lambda}|\nu_{\Lambda}^{free}) \leq \int o(|\Lambda|)\nu_{\Lambda}(dx_{\Lambda}) = o(|\Lambda|) \tag{87}$$

and therefore the relative entropy density

$$i(\nu|\nu^{free}) := \lim_{\Lambda\uparrow\mathbb{Z}^d} \frac{1}{|\Lambda|} I_{\Lambda}(\nu_{\Lambda}|\nu_{\Lambda}^{free}) = 0.$$

□

Corollary 3.1 *The proof of Theorem* 3.1 *provides a constructive way to obtain a solution of the system* (1) *on a small time interval as limit* (*in terms of cluster expansions*) *of finite-dimensional approximations*, *whose existence* (*and uniqueness*) *is ensured by the assumption* (*A*2) (*see Theorem* 11.2 *in* [16]).

Appendix

In this section we want to show that the assumptions on the drift are satisfied for the presented class of examples.

A.1 Example 2.1: Markovian Drift

We will check the condition (A3) when the drift is Markovian. First of all we compute the time-reversal of the functional in X

$$\int_0^t b_0(s, X(s))dX_0(s) - \frac{1}{2}\int_0^t b_0^2(s, X(s))ds =: I_t(X) - \frac{1}{2}\int_0^t b_0^2(s, X(s))ds, \tag{88}$$

where the stochastic integral part $I_t(X)$ is defined as

$$I_t(X) = \int_0^t b_0(s, X(s))dX_0(s) = \lim_{\substack{n\to\infty \\ \Delta s\to 0}} \sum_{j=1}^{n} b_0(s_{j-1}, X(s_{j-1}))(X_0(s_j) - X_0(s_{j-1})) \tag{89}$$

with Δs the mesh size and $0 = s_0 < \cdots < s_n = t$ a partition of $[0, t]$. Then the time-reversal of the stochastic integral given in (89) is

$$I_t \circ \theta_t(X) \quad = \quad \lim_{\substack{n\to\infty \\ \Delta s\to 0}} \sum_{j=1}^{n} b_0(s_{j-1}, X(t - s_{j-1}))(X_0(t - s_j) - X_0(t - s_{j-1}))$$

$$
\begin{aligned}
&\overset{r_j:=t-s_{n-j}}{=} -\lim_{\substack{n\to\infty\\ \Delta r\to 0}} \sum_{j=1}^{n} b_0(t-r_j, X(r_j))(X_0(r_j)-X_0(r_{j-1}))\\
&= \lim_{\substack{n\to\infty\\ \Delta r\to 0}} \sum_{j=1}^{n} b_0(t-r_{j-1}, X(r_{j-1}))(X_0(r_j)-X_0(r_{j-1}))\\
&\quad - \lim_{\substack{n\to\infty\\ \Delta r\to 0}} \sum_{j=1}^{n} b_0(t-r_j, X(r_j)) + b_0(r_{j-1}, X(r_{j-1}))\\
&\quad \times (X_0(r_j)-X_0(r_{j-1}))
\end{aligned}
\tag{90}
$$

which is equal to the sum of an Itô integral and twice a Stratonovich integral,

$$
\int_0^t b_0(t-s, X(s))dX_0(s) - 2\int_0^t b_0(t-s, X(s)) \circ dX_0(s). \tag{91}
$$

Note that X_0 is Brownian motion under the measure P^x. So using the Itô-Stratanovich relation (see e.g. Definition 3.13 in [7]), we obtain under P^x

$$
I_t \circ \theta_t(X) = -\int_0^t b_0(t-s, X(s))dX_0(s) - \int_0^t b_0'(t-s, X(s))ds. \tag{92}
$$

The second integral in (88) is an ordinary Riemann-Stieltjes integral. So we obtain

$$
\left(\int_0^t b_0^2(s, \cdot(s))ds\right) \circ \theta_t(X) = \int_0^t b_0^2(t-s, X(s))ds. \tag{93}
$$

Thus, the time-reversal of (88) is equal to

$$
\begin{aligned}
&F_0^t \circ \theta_t(X)\\
&= -\int_0^t b_0(t-s, X(s))dX_0(s) - \int_0^t \left(b_0'(t-s, X(s)) + \frac{1}{2}b_0^2(t-s, X(s))\right)ds.
\end{aligned}
\tag{94}
$$

To obtain the convergence of $\exp|F_0^t \circ \theta_t(X)|$ towards 1 when t tends to 0 in $L^{2p}(P^x)$, since the a.s. convergence is clear, it is enough to prove a uniform bound for $t \in [0, 1]$ in $L^{2p'}$, $p' > p$. Indeed

$$
\begin{aligned}
&\mathbb{E}_{P^x}\left(\exp\left(2p'\left|F_0^t \circ \theta_t(X)\right|\right)\right)\\
&\quad \le e^{p't(\|b_0\|_\infty^2 + 2\|b_0'\|_\infty)}\mathbb{E}_{P^x}\left(\exp\left(2p'\left|\int_0^t b_0(t-s, X(s))dX_0(s)\right|\right)\right).
\end{aligned}
\tag{95}
$$

The first term on the right side is bounded for $t \in [0, 1]$. The second term can be controlled as follows:

$$
\begin{aligned}
&\mathbb{E}_{P^x}\left(\exp\left(2p'\left|\int_0^t b_0(t-s, X(s))dX_0(s)\right|\right)\right)\\
&\quad \le \mathbb{E}_{P^x}\left(\exp\left(2p'\int_0^t b_0(t-s, X(s))dX_0(s)\right)\right)
\end{aligned}
$$

$$+\mathbb{E}_{P^x}\left(\exp\left(-2p'\int_0^t b_0(t-s,X(s))dX_0(s)\right)\right). \tag{96}$$

Since $\exp(2p'\int_0^t b_0(t-s,X(s))dX_0(s)-2p'^2\int_0^t b_0^2(t-s,X(s))ds)$ (resp. $\exp(-2p'\times\int_0^t b_0(t-s,X(s))dX_0(s)-2p'^2\int_0^t b_0^2(t-s,X(s))ds)$) is a P^x-martingale with expectation 1

$$\begin{aligned}&\mathbb{E}_{P^x}\left(\exp\left(2p'\int_0^t b_0(t-s,X(s))dX_0(s)\right)\right)\le e^{2p'^2t\|b_0\|_\infty^2}\quad\text{and}\\&\mathbb{E}_{P^x}\left(\exp\left(-2p'\int_0^t b_0(t-s,X(s))dX_0(s)\right)\right)\le e^{2p'^2t\|b_0\|_\infty^2},\end{aligned} \tag{97}$$

which are bounded uniformly for $t\in[0,1]$ too.

A.2 Examples 2.2 and 2.3: Interaction with Finite Extent in Space and Time

We want now to do explicit computations for the long-memory example $b_i(t,\omega)=\int_0^t\epsilon(s)(\omega_i(s)-\omega_i(0))ds$ with ϵ satisfying (18). The requirement (A2) holds since

$$\begin{aligned}&\left|\int_0^t\epsilon(s)(\omega_0(s)-\omega_0(0))ds-\int_0^t\epsilon(s)(\omega_0'(s)-\omega_0'(0))ds\right|\\&\quad\le 2\int_0^t\epsilon(s)ds\sup_{0\le s\le t}|\omega_0(s)-\omega_0'(s)|.\end{aligned} \tag{98}$$

To prove the condition (A3) we first analyse the stochastic integral term $J_t(X):=\int_0^t b_0(s,X)dX_0(s)$.

$$\begin{aligned}J_t(X)&=\int_0^t\int_0^s\epsilon(r)(X_0(r)-X_0(0))drdX_0(s)\\&=\int_0^t\epsilon(r)\int_r^t dX_0(s)(X_0(r)-X_0(0))dr\\&=\int_0^t\epsilon(r)(X_0(t)-X_0(r))(X_0(r)-X_0(0))dr\end{aligned} \tag{99}$$

(for the interchange of the order of integration, see for example the lecture notes [17]). The integral is now an ordinary Riemann-Stieltjes one. Hence, its time-reversal satisfies

$$\begin{aligned}&J_t\circ\theta_t(X)\\&\quad=\int_0^t\epsilon(r)(X_0(0)-X_0(t-r))(X_0(t-r)-X_0(t))dr\\&\quad=\int_0^t\epsilon(t-r')(X_0(t)-X_0(r'))(X_0(r')-X_0(0))dr'.\end{aligned} \tag{100}$$

Similar computations lead us to the time-reversal of the functional

$$X\mapsto\left(\int_0^t\left(\int_0^s\epsilon(r)(X_0(r)-X_0(0))dr\right)^2ds\right).$$

One obtains

$$\int_0^t \left(\int_0^s \epsilon(t-r)(X_0(r) - X_0(0))dr \right)^2 ds.$$

Thus

$$F_0^t \circ \theta_t(X) = \int_0^t \epsilon(t-s)(X_0(t) - X_0(s))(X_0(s) - X_0(0))ds - \frac{1}{2} \int_0^t \left(\int_0^s \epsilon(t-r)(X_0(r) - X_0(0))dr \right)^2 ds. \tag{101}$$

As in the above Example A.1 the convergence of $\exp|F_0^t \circ \theta_t(X)|$ in $\mathbb{L}_{2p}(P^x)$ for $t \to 0$ is a direct consequence of a uniform bound for $t \in [0, 1]$ in $L^{2p'}$, $p' > p$, which we now prove.

$$\begin{aligned}
&\mathbb{E}_{P^x}\big(\exp\big(2p' \, |F_0^t \circ \theta_t(X)|\big)\big) \\
&\quad \le \mathbb{E}_{P^x}\big(\exp\big(p'\varepsilon(t)(2 + t\varepsilon(t)) \sup_{s \le t}\big[X_0(s) - X_0(0)\big]^2\big)\big) \\
&\quad \le \mathbb{E}_{P_0^{x_0}}\big(\sup_{s \le t}\big[\exp\big(X(s) - X(0)\big)^2\big]^{p'\varepsilon(t)(2+t\varepsilon(t))}\big) \\
&\quad \le \mathbb{E}_{P_0^{x_0}}\big(\sup_{s \le t}\big[\exp\big(X(s) - X(0)\big)^2\big]^c\big)
\end{aligned} \tag{102}$$

where $\varepsilon(t) =: \int_0^t \epsilon(s)ds$ and $c = p'\varepsilon(1)(2 + \varepsilon(1))$. Since $X(t)$ is a Brownian motion w.r.t. $P_0^{x_0}$, we can apply Doob's inequality and then obtain

$$\begin{aligned}
&\mathbb{E}_{P^x}\big(\exp\big(2p'|F_0^t \circ \theta_t(X)|\big)\big) \\
&\quad \le \left(\frac{c}{c-1}\right)^c \mathbb{E}_{P_0^{x_0}}\big(\exp 2c|X(t) - X(0)|\big) \\
&\quad \le \left(\frac{c}{c-1}\right)^c \mathbb{E}\big(\exp(2c\sqrt{t}|Z|)\big) \\
&\quad \le \left(\frac{c}{c-1}\right)^c \mathbb{E}\big(\exp(2c|Z|)\big) < +\infty
\end{aligned} \tag{103}$$

where Z is a standard Gaussian variable. The proof of (A3) is now completed for the Example 2.2.

Example 2.3 can be treated in a very similar way, we leave the straightforward details here to the reader.

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